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The Deregulated Electricity Market Viewed as a Bilevel Programming Problem

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Abstract. In this paper, we present a bilevel programming formulation of a deregulated electricity market. By examining the electricity market in this format, we achieve two things. First, the relation of the deregulated electricity market to general economic models that can be formulated as bilevel programming problems (e.g. Stackelberg leader-follower games and principal-agency models) becomes clear. Secondly, it provides an explanation of the reason why the so-called "folk theorems" can be proven to be false for electricity networks. The interpretation of the deregulated electricity market as a bilevel program also indicates the magnitude of the error that can be made if the electricity market model studied does not take into account the physical constraints of the electric grid, or oversimplifies the electricity network to a radial network.

Key words: Bilevel programming; Energy markets; Power transmission

1. Introduction

Within the last decades there has taken place a restructuring of the electricity industries all over the world. In the Nordic countries, there is now a common market for electricity, and energy is traded vigorously both in the physical and financial markets of Nord Pool, the Nordic power exchange. The goal of the deregulation has been to achieve overall short run and long run efficiency through competition on the supply and demand side, and through the efficient pricing of transmission, which is still considered to be a natural monopoly function and regulated accordingly. Compared to the regulated state of the world, there has taken place an unbundling of the service, and both industrial consumers as well as households can choose energy provider independent of the local distributor.

An essential part of the regulation and electricity market design is a system for managing congestions in the transmission network, due to for instance thermal capacity limitations that restrict the power flows. The classic benchmark for the optimal utilization of the power system is the optimal economic dispatch, providing optimal nodal prices that give the value of power in every single node or location in the grid (Schweppe *et al.* 1988). Due to capacity constraints (as well as losses), the differences between the various area prices can be of considerable size, implying that the management of congestion may affect the short-term formation of prices to a substantial extent. This also explains the ample interest that the various agents in the industry show towards transmission pricing and practical methods for congestion management.

In the literature Wu *et al.* (1996) give counter-examples to a number of propositions regarding the characteristics of optimal nodal prices, which at first sight, without any specific knowledge of power networks, seem quite intuitive. Among the "folk theorems" that are proven false are

(1) Uncongested lines do not receive congestion rents (defined through nodal price differences);
(2) In an efficient allocation power can only flow from nodes with lower prices to nodes with higher prices; and
(3) Strengthening transmission lines or building additional lines increases transmission capacity.

It is argued that these assertions stem from the incorrect analogy between power transmission and the transportation of goods. Economic analyses of the transportation of goods can be found already in the classical works on *spatial price equilibrium* by Enke (1951) and Samuleson (1952), and the problems are, at least conceptually, very similar to the optimal dispatch problem of electric power distribution.[1]

While appealing to economic intuition, this paper intends to give one possible explanation of the foundation for the difference between markets that are based on power transmission networks and spatial markets based on simpler models for transportation of goods, like commodity flows or transportation problems. The paper is organized as follows: In Section 2, the optimal economic dispatch problem is presented, in Section 3 a bilevel programming formulation is derived for the direct current (DC) analogy, whereas a similar construct and interpretation is given in Section 4 for the "DC" approximation of alternating current (AC) power flows. In Section 5, some implications of the bilevel structure of the optimal dispatch problem are discussed, and finally, conclusions are given in Section 6.

2. Optimal Economic Dispatch

In general, power production and consumption involve both real and reactive power, where real power represents the consumption of energy, and

[1] The spatial price equilibrium model can be phrased as follows. Buyers and sellers of a commodity are located at the nodes of a transportation network, and the issue is to determine simultaneously the quantities supplied and demanded at each node, the local (nodal) prices at which the commodity is bought and sold, and the commodity flows between pairs of nodes.

reactive power is needed for system operation, for instance voltage control. Consequently, power in AC systems is often described by complex numbers, where complex power S consists of a real power part P and a reactive power part Q.

In the following, let $B_i(S_i^d)$ be the benefit from consuming complex power $S_i^d = P_i^d + jQ_i^d$ and $C_i(S_i^s)$ the cost of producing $S_i^s = P_i^s + jQ_i^s$ in node i ($j^2 = -1$). A general formulation of the optimal dispatch problem, taking into account thermal capacity limits, is then given by problem (1)–(7) (Wangensteen *et al.* 1995):

$$\max \sum_i [B_i(S_i^d) - C_i(S_i^s)] \tag{1}$$

$$\text{s.t.} \quad S_i = S_i^s - S_i^d \quad \forall i \tag{2}$$

$$S_i = V_i \cdot I_i^* \quad \forall i \tag{3}$$

$$S_{ik} = V_i \cdot I_{ik}^* \quad \forall ik \tag{4}$$

$$|S_{ik}| \le C_{ik} \quad \forall ik \tag{5}$$

$$I_i = \sum_{k \ne i} I_{ik} \quad \forall i \tag{6}$$

$$I_{ik} = Y_{ik}(V_i - V_k) \quad \forall ik \tag{7}$$

The objective function (1) maximizes social surplus, while summing benefits and withdrawing cost over all the nodes. Equation (2) defines net injection $S_i = P_i + jQ_i$ in every node, and (3) and (4) relate complex power to complex voltage V_i and the conjugates of complex node and line currents I_i and I_{ik}, respectively. Inequalities (5) represent the thermal capacity constraints, which are stated in terms of limits C_{ik} on the magnitude of apparent power, $|S_{ik}| = \sqrt{P_{ik}^2 + Q_{ik}^2}$. Equations (6) represent Kirchhoff's junction rule and (7) Ohm's law with Kirchhoff's loop rule incorporated, Y_{ik} being the admittance of line ik.

In general, this optimal dispatch problem is non-convex, but under normal operation, simpler models can be used to approximate the general expressions, in order to highlight specific elements of the operations of the exceedingly complex power systems. In the next sections we will highlight the bilevel nature of the optimal dispatch problem, which solution the deregulated electricity market is to replicate, by means of the DC model and the ''DC'' approximation of AC power flows.

3. A Bilevel Programming Formulation for the DC Analogy

It is well known (since the work of Kirchhoff and Maxwell in the 19th century) that the physical equilibrium of electric networks can be described in terms of minimization of total power-losses, i.e., the electric current follows the path of least resistance. To simplify, consider now a DC model, where all power flows, voltages and currents of problem (1)–(7) are real numbers. Given node currents I_i, optimal line currents I_{ik} are obtained by solving the following convex flow problem (see for instance Dembo *et al.* 1989):

$$\min \frac{1}{2}\sum r_{ik} I_{ik}^2 \tag{8}$$

$$\text{s.t. } I_i = \sum_{k \neq i} I_{ik} \quad \forall i \tag{9}$$

where r_{ik} is the resistance of line ik.

Introducing dual variables V_i of equation (9), the Lagrangian can be written

$$\Phi = \frac{1}{2}\sum_{ik} r_{ik} I_{ik}^2 + \sum_i \left(I_i - \sum_{k \neq i} I_{ik} \right) . V_i \tag{10}$$

with first order conditions

$$\frac{\partial \Phi}{\partial I_{ik}} = r_{ik} I_{ik} - V_i + V_k = 0 \quad \forall ik \tag{11}$$

and

$$\frac{\partial \Phi}{\partial V_i} = I_i - \sum_{k \neq i} I_{ik} = 0 \quad \forall i. \tag{12}$$

Condition (11) implies

$$I_{ik} = \frac{V_i - V_k}{r_{ik}} = Y_{ik}(V_i - V_k) \quad \forall ik \tag{13}$$

since admittance $Y_{ik} = 1/r_{ik}$ in a DC network. I.e. the first order conditions of problem (8)–(9) correspond to equations (6) and (7). This means that we can reformulate the optimal dispatch problem (assuming a DC network with real power only, i.e., $S_i = P_i$ to:

$$\max_{P_i^s, P_i^d, I_i} \sum_i [B_i(P_i^d) - C_i(P_i^s)] \tag{P1}$$

$$\begin{aligned} \text{s.t. } P_i &= P_i^s - P_i^d \quad \forall i \\ P_i &= V_i I_i \quad \forall i \\ P_{ik} &= V_i I_{ik} \quad \forall ik \\ P_{ik} &\leq C_{ik} \quad \forall ik \end{aligned}$$

and given $I_i \; \forall i, I_{ik}$ is implicitly defined by,

$$\min \frac{1}{2}\sum r_{ik} I_{ik}^2 \tag{P2}$$

$$\text{s.t. } I_i = \sum_{k \neq i} I_{ik} \quad \forall i$$

which provides also the dual variables V_i. In this formulation it is evident that the first level, P1, sets the node currents, and the ''agents'', the electrons, react on this by following the path of least resistance. Hence, in economic modeling terms this, represented by P2, is the behavioral assumption made upon the ''agents''.

4. A Bilevel Programming Formulation for the ''DC'' Approximation

In an AC network, real power over line *ik* is often given by the formulae

$$P_{ik} = -G_{ik} V_i^2 + G_{ik} V_i V_k \cos(\delta_i - \delta_k) + B_{ik} V_i V_k \sin(\delta_i - \delta_k) \tag{14}$$

where G_{ik} is the *conductance* of line *ik*, B_{ik} is the *susceptance*, V_i is the voltage magnitude of node *i*, and $\delta_i - \delta_k$ is the phase angle difference between nodes *i* and *k* (see for instance Wood and Wollenberg 1996). The ''DC'' approximation of the real power flows in an AC network assumes that line resistance, r_{ik}, and line reactance, x_{ik} are such that $r_{ik} << x_{ik}$. Since the conductance is given by $G_{ik} = -r_{ik}/(r_{ik}^2 + x_{ik}^2)$, the two first terms on the right hand side of equation (14) can be approximated by zero. Assuming voltage magnitudes equal to 1, and small phase angle differences, such that $\sin(\delta_i - \delta_k) \approx (\delta_i - \delta_k)$, we have the following approximation for the real power flows of the lines

$$P_{ik} \approx B_{ik}(\delta_i - \delta_k) = \frac{x_{ik}}{r_{ik}^2 + x_{ik}^2}(\delta_i - \delta_k) \quad \forall ik \tag{15}$$

This means that the optimal economic dispatch problem under the ''DC'' approximation of the power flows can be formulated as follows:

$$\max_{P_i^s, P_i^d} \sum_i [B_i(P_i^d) - C_i(P_i^s)] \tag{16}$$

$$\text{s.t. } P_i = P_i^s - P_i^d \quad \forall i \tag{17}$$

$$P_i = \sum_{i \neq k} P_{ik} \quad \forall i \tag{18}$$

$$P_{ik} = B_{ik}(\delta_i - \delta_k) \quad \forall ik \tag{19}$$

$$P_{ik} \leq C_{ik} \quad \forall ik \tag{20}$$

By using the Lagrangian, we find that (18) and (19) are the first order necessary conditions of the optimization problem

$$\min \frac{1}{2} \sum \frac{r_{ik}^2 + x_{ik}^2}{x_{ik}} P_{ik}^2 \tag{21}$$

$$\text{s.t. } P_i = \sum_{k \neq i} P_{ik} \quad \forall i \tag{22}$$

with dual variables δ_i for equations (22). That means that also in the case of the "DC" approximation, the optimal economic dispatch problem can be formulated as a bilevel programming problem, namely

$$\max_{P_i^s, P_i^d} \sum_i [B_i(P_i^d) - C_i(P_i^s)] \tag{P3}$$

$$\text{s.t. } P_i = P_i^s - P_i^d \quad \forall i$$

$$P_i = \sum_{k \neq i} P_{ik} \quad \forall i$$

$$P_{ik} \leq C_{ik} \quad \forall ik$$

where P_{ik} is determined by

$$\min \frac{1}{2} \sum \frac{r_{ik}^2 + x_{ik}^2}{x_{ik}} P_{ik}^2 \tag{P4}$$

$$\text{s.t. } P_i = \sum_{k \neq i} P_{ik} \quad \forall i$$

The question is now whether the lower level problem P4 can be given a similar interpretation as P2. Under the "DC" approximation, the focus is on real power, and real power losses over a line, though assumed to be negligible, are approximated by $r_{ik} P_{ik}^2$ (Schweppe *et al.* 1988). However, there is still reactive power loading and reactive losses, which could be estimated under the same assumptions. Chao and Peck (1996) use the following expression for reactive power over line ik[2]

$$Q_{ik} = B_{ik} V_i^2 + G_{ik} V_i V_k \sin(\delta_i - \delta_k) - B_{ik} V_i V_k \sin(\delta_i - \delta_k)$$

The reactive loss over line ik, l_{ik}^Q, is then equal to

$$l_{ik}^Q = Q_{ik} + Q_{ki} = B_{ik} \left[V_i^2 + V_k^2 - 2 V_i V_k \cos(\delta_i - \delta_k) \right]$$

Utilizing the assumptions of the "DC" approximation, and the second order Taylor approximation for $\cos(\delta_i - \delta_k)$, reactive losses can be approximated by

$$\begin{aligned} l_{ik}^{Q} &= B_{ik}\left[V_i^2 + V_k^2 - 2V_iV_k \cos\,(\delta_i - \delta_k)\right] \\ &\approx B_{ik}\left[1 + 1 - 2\left(1 - \frac{(\delta_i - \delta_k)^2}{2}\right)\right] \\ &= B_{ik}(\delta_i - \delta_k)^2 = B_{ik}\left(\frac{P_{ik}}{B_{ik}}\right)^2 \\ &= \frac{r_{ik}^2 + x_{ik}^2}{x_{ik}} P_{ik}^2 \approx x_{ik}P_{ik}^2 \end{aligned}$$

The real power approximation is robust with respect to small relaxations of the assumptions, but this is not the case for the reactive power flows. However, at nominal voltages and small angle differences, the approximation applies, even if it is not robust. This leaves the subproblem P4 with the interpretation of minimizing reactive losses under the maintained assumption that voltage magnitudes are equal to unity, and the voltage angles are small so that the real power losses can be ignored.[3]

5. Discussion and Implications

Problems P1–P2 and P3–P4 fit into the framework of bilevel programs that are discussed in Kolstad (1985). Thus, the optimal dispatch problem can be seen as a bilevel program consisting of an *upper level* program, which is the social maximization problem, P1 or P3, and a *lower level* program or *behavioral* problem, P2 or P4, which determines line currents/power flows and, as a byproduct, voltages. The intention of these bilevel constructions is to reveal the structure of the optimal economic dispatch problem, not to indicate how it should be solved. In general, the problem is highly non-linear and non-convex with interdependencies between the variables. According to the classification of Kolstad (1985), formulations like (1)–(7) or (16)–(20) can be understood to arise after applying a *Kuhn–Tucker–Karush*-method to the bilevel program, transforming the behavioral problem into Kuhn–Tucker–Karush necessary conditions for optimality, and solving the resulting problem is equivalent to solving the original bilevel program.

A number of economic problems can be interpreted as bilevel programs. For instance, a *Stackelberg leader-follower* game can be viewed as a bilevel program with the leader's problem corresponding to P1/P3 and the follower's problem corresponding to P2/P4 (Kolstad 1985, Migdalas and Pardalos

[2] Siddiqi and Baughman (1995) add the term $-1/2\beta_{ik}V_i^2$, where β_{ik} is the parameter for the shunt susceptance of line ik.

[3] With the extra term of footnote 2, we must add $-\beta_{ik}(V_i^2 + V_k^2)$ to the expression for the reactive loss. However, this term is a constant, which would not be affected by the minimization, and thus, the interpretation still applies.

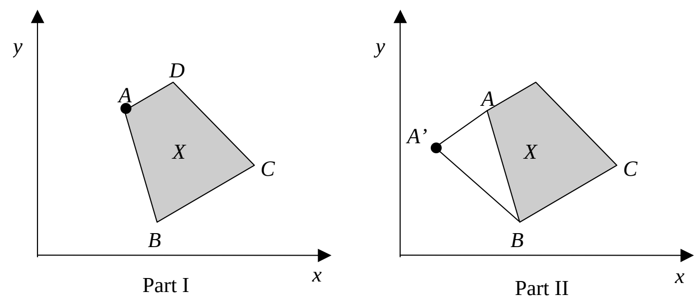
1993, and Vicente and Calamai 1994). In this type of model, the follower chooses his strategy in full knowledge of the leader's decision, a fact that the leader takes into consideration when determining his own actions. Similarly, principal-agent problems can be interpreted in the same manner, as the principal takes into account the behavior of the agent acting in his own self interest (modeled through P2/P4) when solving the upper level program P1/P3.

Returning to the optimal dispatch problem of electrical networks, and the discussion of Wu *et al.* (1996) concerning the incorrect analogy between power transmission and transportation of goods, constraint (6), which is Kirchhoff's junction rule, is normally accounted for in most transportation models. However, if one is to disregard Kirchhoff's loop rule in the analysis, thus assuming power is routable, the error made may be of the same order as ignoring the behavior of the followers in a Stackelberg leader-follower game or the behavior of the agents in a principal-agent setting. In relation to the bilevel programs formulated in this paper, this would correspond to taking into account only a subset of the necessary conditions for the lower level problems P2 and P4. The loop flow phenomenon, represented by the way flow distributes over the grid, and given by equations (13) and (19) for the DC analogy and "DC" approximation respectively, thus may be regarded as a consequence of some lower level optimization problem within the welfare maximization problem.

The bilevel constructions of this paper may give intuition to some of the "peculiarities" of electrical networks. Let us for instance consider the very simple two-variable bilevel program given by

$$\begin{aligned} &\max_x \; y \\ &\text{s.t. } (x, y) \in X \\ &\quad\;\; \min\,(y|x) \end{aligned}$$

The upper level program represents a principal that maximizes y by choosing x. In addition to the constraint that the solution must belong to the feasible set X, the principal must also take into account the agent that minimizes y for given x. The problem and its solution can be illustrated in the following figure:



In Part I, the principal's optimal solution is to choose x corresponding to point A. This is so because the behavior of the agent, restricting solutions to line segments A–B–C, makes point D unattainable even if it belongs to the feasible set X. In Part II, we consider an expansion of the feasible set X. Due to the bilevel nature of the problem at hand, this will actually reduce the value of the optimal solution for the principal, as the agent now enforces solutions along A′–B–C. The possibility of having this type of effect is rather obvious in a programming problem with a bilevel structure, but much more difficult to spot when for instance the lower level problem has been replaced by its first order conditions for optimality.

Despite obvious similarities between the operation of the power market and spatial price equilibrium models, focusing on the physical equilibrium of a power network leads to the awareness that one should rather have in mind something similar to *traffic equilibrium* problems as the underlying network model when investigating power markets. In power networks, strengthening a line may lead to reduced transmission capacity and/or reduce social surplus in optimal dispatch, which is an analogy to the famous Braess' paradox (1968) in traffic equilibrium networks. Also the same *non-cooperative* phenomenon is recognized in communication networks, as is evident from the works of for instance, MacKie-Mason and Varian (1995), Shenker (1995), Shenker *et al.* (1996), Korilis *et al.* (1997a, b) and Gupta *et al.* (1997). The phenomenon is also discussed in Calvert and Keady (1993) and Bean *et al.* (1997).

6. Conclusions

Even in a relatively well developed power market as the Nordic, there is a continuously going discussion of the regulation of the transportation function and the specifics of the market design, especially related to congestion management and system security in general. For decision-makers in the power market, especially those concerned with market design and regulation, but also others that operate in the power market, it is very important to be aware of the "peculiarities" that may result in power transmission.

Viewing the optimal dispatch problem as a bilevel mathematical program with interacting physical and economic equilibria may make it easier for an economist to understand the difference between classical spatial equilibrium models and equilibria in power networks. It also provides a way to get an understanding of the magnitude of the error that can be made by simplifying the network description by disregarding Kirchhoff's loop rule or by simplifying the network description to a radial network.

As a by-effect, the formulation can lead to new ideas regarding optimal transmission pricing in a decentralized electricity market. For instance, instead of (or additional to) checking if a market equilibrium is physically feasible, one could check whether a physical equilibrium is economically viable. It can also be fruitful to have this formulation in mind when simplifying an electricity network into a virtual radial network to be used in aggregate electricity market models. Whether these are interesting approaches, and how they could be used in a practical procedure, is a topic for future investigation.

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References

1. Bean, N.G., Kelly, F.P. and Taylor, P.G. (1997), Braess' paradox in a loss network. *Journal of Applied Probability* 34, 155–159.
2. Braess, D. (1968), Über ein Paradoxon aus der Verkehrsplannung. *Unternehmensforschung* 12, 258–268.
3. Calvert, B. and Keady, G. (1993), Braess's paradox and power-law nonlinearities in networks. *Journal of the Australian Mathematical Society, Series B Applied Mathematics* 35, 1–22.
4. Chao, H.-P. and Peck, S. (1996), A market mechanism for electric power transmission. *Journal of Regulatory Economics* 10, 25–59.
5. Dembo, R.S., Mulvey, J.M. and Zenios, S.A. (1989), Large-scale nonlinear network models and their application. *Operations Research* 37, 353–372.
6. Enke, S. (1951), Equilibrium among spatially separated markets: solution by electric analogue. *Econometrica* 19, 40–47.
7. Gupta, A., Stahl, D.O. and Whinston, A. (1997), A stochastic equilibrium model of internet pricing. *Journal of Economic Dynamics and Control* 21, 697–722.
8. Kolstad, C.D. (1985), A review of the literature on bi-level mathematical programming, *Technical Report LA-10284-MS, US-32,* Los Alamos National Laboratory.
9. Korilis, Y.A., Lazar, A.A. and Orda, A. (1997a), Achieving network optima using stackelberg routing strategies. *IEEE/ACM Transactions on Networking* 5, 161–173.
10. Korilis, Y.A., Lazar, A.A. and Orda, A. (1997b), Avoiding the Braess paradox in noncooperative networks. *Journal of Applied Probability* 36, 211–222.
11. MacKie-Mason, J.K. and Varian, H.R. (1995), Pricing congestible network resources. *IEEE Journal on Selected Areas in Communications* 13, 1141–1149.
12. Migdalas, A. and Pardalos, P.M. (1993), Editorial: Hierarchical and bilevel programming. *Journal of Global Optimization* 8, 209–215.
13. Samuelson, P.A. (1952), Spatial Price Equilibrium and Linear Programming. *The American Economic Review* 42, 283–303.
14. Schweppe, F.C., Caramanis, M.C., Tabors, R.D. and Bohn, R.E. (1988), *Spot Pricing of Electricity*, Kluwer Academic Publishers.

15. Shenker, S. (1995), Making greed work in networks: a game-theoretic analysis of switch service disciplines. *IEEE/ACM Transactions on Networking* 3, 819–831.
16. Shenker, S., Clark, D., Estrin, D. and Herzog, S. (1996), Pricing in computer networks: reshaping the research agenda. *Telecommunications Policy* 20, 183–201.
17. Siddiqi, S.N. and Baughman, M.L. (1995), Value-based transmission planning and the effects of network models. *IEEE Transactions on Power Systems* 10, 1835–1842.
18. Vicente, L.N. and Calamai, P.H. (1994), Bilevel and multilevel programming: a bibliography review. *Journal of Global Optimization* 5, 291–306.
19. Wangensteen, I., Holtan, J.A., Fosso, O.B., Faanes, H.-H. and Hornnes, K.S. (1995), Avbildning av korttids grensekostnad i sentralnettstariffen. *Teknisk Rapport F4313*, EFI Sintef gruppen, Trondheim.
20. Wood, A.J., and Wollenberg B.F. (1996), *Power Generation, Operation, and Control*, Second Edition, John Wiley & Sons.
21. Wu, F., Varaiya, P. Spiller, P., and Oren, S. (1996), Folk theorems on transmission access: proofs and counterexamples. *Journal of Regulatory Economics* 10, 5–23.